Symbols

Special sets

$\mathbb{B} = \{0, 1\}$, $\mathbb{B}^n$
$\mathfrak{M}_{m,n}$ [$m \times n$ matrices]
$\mathbb{N}$ [integers ≥ 0]
$\mathbb{P}$ [positive integers]
$\mathbb{Q}$ [rationals]
$\mathbb{R}$ [real numbers]
Σ^*, Σ
$\mathbb{Z}$ [all integers]
$\mathbb{Z}(p)$, $[n]_p$
$[a, b]$, (a, b), etc.

Set notation

$a \in A$, $a \notin A$
A^c [complement]
$A \setminus B$
$A \cup B$ [union]
$A \cap B$ [intersection]
$A \oplus B$ [symmetric difference]
$\mathcal{P}(S)$ [power set]
(s, t), $(s_1, \ldots, s_n)$
$S \times T$, $S^2 = S \times S$
$S_1 \times S_2 \times \cdots \times S_n$, S^n
$\bigcup_{k=0}^{\infty} A_k$, $\bigcap_{k=0}^{\infty} A_k$, $\bigcup_{k=0}^{m} A_k$, etc.
$T \subseteq S$
$T \subset S$
$\emptyset$ [empty set]

Functions

χ_A [characteristic function]
$\mathrm{Dom}(f)$ [domain of f]
$f \circ g$ [composition]
$f(A)$
$f: S \to T$
f^{-1} [inverse function]
$f^{\leftarrow}(B)$, $f^{\leftarrow}(y)$
$\mathrm{FUN}(S, T)$
$\mathrm{Graph}(f)$
$\mathrm{Im}(f)$ [image of f]
$\log x$, $\ln x$
(s_n) [sequence]

Σ^*

λ [empty word or list]
$\mathrm{length}(w)$
Σ [alphabet]
Σ^* [words]
Σ^k [words of length k]
$\overleftarrow{w}$ [reversal]

Miscellany

$a := b$
$\lfloor a \rfloor$ [floor]
$\lceil a \rceil$ [ceiling]
$n!$ [factorial]
$|x|$ [absolute value]
$a | b$ [a divides b]
$m \equiv n \pmod{p}$
$m +_p n$, $m *_p n$
$m * n$ [product]
$m \wedge n$ [m^n]
$n \text{ DIV } p$, $n \text{ MOD } p$
$\gcd(m, n)$
$\mathrm{lcm}(m, n)$
$\max\{a, b\}$
$\min\{a, b\}$
$O(n^2)$, $O(n \log n)$, etc.
$\Theta(n^2)$, $\Theta(n \log n)$, etc.
$\prod$ [product]
$\sum$ [sum]
∞, $-\infty$
$\blacksquare$

Logic

$\neg p$ [negation]
$p | q$ [Sheffer stroke]
$p \wedge q$, $p \vee q$ [and, or]
$p \to q$ [implication]
$P \Longrightarrow Q$
$p \leftrightarrow q$ [biconditional]
$P \Longleftrightarrow Q$
$p \oplus q$ [exclusive or]
$\mathbf{1}$ [tautology]
$\mathbf{0}$ [contradiction]
$\forall$, $\exists$
$\exists!$
$\therefore$

Matrices

$\mathbf{A} = [a_{jk}]$
$\mathbf{A}[j,k] = a_{jk}$
$\mathbf{A}^{-1}$ [inverse of $\mathbf{A}$]
$\mathbf{A}^T$ [transpose of $\mathbf{A}$]
$\mathbf{A} + \mathbf{B}$ [sum]
$\mathbf{AB}$ [product]
$\mathbf{A}_1 * \mathbf{A}_2$ [Boolean product]
$c\mathbf{A}$ [scalar product]
$-\mathbf{A}$ [negative of $\mathbf{A}$]
$\mathbf{I}_n$ [$n \times n$ identity matrix]
$\mathbf{M}_R$
$\mathfrak{M}_{m,n}$ [$m \times n$ matrices]
$\mathbf{0}$ [zero matrix]

Relations

R_f [for a function f]
$R^{\leftarrow}$ [converse relation]
$f^{\leftarrow}$ [as a relation]
$\sim$ [equivalence]
$[s]$ [equivalence class]
$[S]$
$\preceq$, $\prec$, $(S, \preceq)$
$\max(S)$, $\min(S)$
$\text{lub}(S)$, $\text{glb}(S)$
$x \vee y$, $x \wedge y$
$\text{FUN}(S, T)$
$\preceq$ [on $\text{FUN}(S, T)$]
$\preceq^k$ [filing order]
$\preceq_{LL}$ [lenlex order]
$\preceq_L$ [lexicographic order]
E [equality relation]
$R_1 R_2 = R_2 \circ R_1$
$\mathbf{A}_1 * \mathbf{A}_2$ [Boolean product]
R^n, R^0
$\mathbf{A}_1 \leq \mathbf{A}_2$ [Boolean matrices]
$\mathbf{A}_1 \vee \mathbf{A}_2$, $\mathbf{A}_1 \wedge \mathbf{A}_2$
$r(R)$, $s(R)$, $t(R)$
$\mathbf{r}(\mathbf{A})$, $\mathbf{s}(\mathbf{A})$, $\mathbf{t}(\mathbf{A})$

Algebraic Systems

$x \vee y$, $x \wedge y$
x' [complement]
$x \leq y$
0, 1
$\mathbb{B}$, $\mathbb{B}^n$
$\text{BOOL}(n)$ [Boolean functions]
$\text{PERM}(X)$ [permutations]
S_n [symmetric group]
$\langle g \rangle$, $\langle A \rangle$ [group generated]
$G(x) = \{g(x) : g \in G\}$ [orbit]
$\text{AUT}(D)$ [automorphisms]
$\text{FIX}_G(x) = \{g \in G : g(x) = x\}$
$\text{FIX}_X(g) = \{x \in X : g(x) = x\}$
$C(k)$ [colorings]
g^*
g^{-1} [inverse of g]
gH, Hg [cosets]
G/H [left cosets]
A^+ [semigroup generated]
R/I [as a ring]

Graphs and trees

$V(G)$, $E(G)$
$\deg(v)$, $\text{indeg}(v)$, etc.
$D_k(G)$
F [float time]
$G \setminus \{e\}$
$G \simeq H$ [isomorphic graphs]
K_n [complete graph]
$K_{m,n}$
M [max-weight]
$\mathbf{M}_R$
$R(v)$
$\text{SUCC}(v)$, $\text{ACC}(v)$
T_r, T_v [rooted trees]
W [weight]
W^* [min-weight]
$W(G)$ [weight of graph]
$W(T)$ [weight of tree]

Counting and probability

$\binom{n}{r}$
$\binom{n}{n_1\ n_2\ \cdots\ n_k}$
$P(n, r)$
$|S|$
Ω [sample space]
$E(X) = \mu$ [expectation]
F_X [cdf]
$P(E)$ [probability of E]
$P(E|S)$ [conditional probability]
$P(X = 2)$, etc.
σ [standard deviation]
$V(X) = \sigma^2$
$\widetilde{X}$, $\widetilde{F}$ [normalizations]
Φ [Gaussian normal]

Some Special Sets

1. List five elements in each of the following sets.
   (a) $\{n \in \mathbb{N} : n \text{ is divisible by } 5\}$
   (b) $\{2n+1 : n \in \mathbb{P}\}$
   (c) $\mathcal{P}(\{1, 2, 3, 4, 5\})$
   (d) $\{2^n : n \in \mathbb{N}\}$
   (e) $\{1/n : n \in \mathbb{P}\}$
   (f) $\{r \in \mathbb{Q} : 0 < r < 1\}$
   (g) $\{n \in \mathbb{N} : n+1 \text{ is prime}\}$
2. List the elements in the following sets.
   (a) $\{1/n : n = 1, 2, 3, 4\}$
   (b) $\{n^2 - n : n = 0, 1, 2, 3, 4\}$
   (c) $\{1/n^2 : n \in \mathbb{P},\ n \text{ is even and } n < 11\}$
   (d) $\{2 + (-1)^n : n \in \mathbb{N}\}$
3. List five elements in each of the following sets.
   (a) Σ^* where $\Sigma = \{a, b, c\}$
   (b) $\{w \in \Sigma^* : \text{length}(w) \le 2\}$ where $\Sigma = \{a, b\}$
   (c) $\{w \in \Sigma^* : \text{length}(w) = 4\}$ where $\Sigma = \{a, b\}$
   Which sets above contain the empty word λ?
4. Determine the following sets, i.e., list their elements if they are nonempty, and write Ø if they are empty.
   (a) $\{n \in \mathbb{N} : n^2 = 9\}$ (b) $\{n \in \mathbb{Z} : n^2 = 9\}$
   (c) $\{x \in \mathbb{R} : x^2 = 9\}$ (d) $\{n \in \mathbb{N} : 3 < n < 7\}$
   (e) $\{n \in \mathbb{Z} : 3 < |n| < 7\}$ (f) $\{x \in \mathbb{R} : x^2 < 0\}$
5. Repeat Exercise 4 for the following sets.
   (a) $\{n \in \mathbb{N} : n^2 = 3\}$
   (b) $\{x \in \mathbb{Q} : x^2 = 3\}$
   (c) $\{x \in \mathbb{R} : x < 1 \text{ and } x \ge 2\}$
   (d) $\{3n+1 : n \in \mathbb{N} \text{ and } n \le 6\}$
   (e) $\{n \in \mathbb{P} : n \text{ is prime and } n \le 15\}$ [Remember, 1 isn't prime.]
6. Repeat Exercise 4 for the following sets.
   (a) $\{n \in \mathbb{N} : n|12\}$ (b) $\{n \in \mathbb{N} : n^2 + 1 = 0\}$
   (c) $\{n \in \mathbb{N} : \lfloor \frac{n}{3} \rfloor = 8\}$ (d) $\{n \in \mathbb{N} : \lceil \frac{n}{2} \rceil = 8\}$
7. Let $A = \{n \in \mathbb{N} : n \le 20\}$. Determine the following sets, i.e., list their elements if they are nonempty, and write Ø if they are empty.
   (a) $\{n \in A : 4|n\}$
   (b) $\{n \in A : n|4\}$
   (c) $\{n \in A : \max\{n, 4\} = 4\}$
   (d) $\{n \in A : \max\{n, 14\} = n\}$
8. How many elements are there in the following sets? Write ∞ if the set is infinite.
   (a) $\{n \in \mathbb{N} : n^2 = 2\}$
   (b) $\{n \in \mathbb{Z} : 0 \le n \le 73\}$
   (c) $\{n \in \mathbb{Z} : 5 \le |n| \le 73\}$
   (d) $\{n \in \mathbb{Z} : 5 < n < 73\}$
   (e) $\{n \in \mathbb{Z} : n \text{ is even and } |n| \le 73\}$
   (f) $\{x \in \mathbb{Q} : 0 \le x \le 73\}$
   (g) $\{x \in \mathbb{Q} : x^2 = 2\}$
   (h) $\{x \in \mathbb{R} : x^2 = 2\}$
9. Repeat Exercise 8 for the following sets.
   (a) $\{x \in \mathbb{R} : 0.99 < x < 1.00\}$
   (b) $\mathcal{P}(\{0, 1, 2, 3\})$
   (c) $\mathcal{P}(\mathbb{N})$
   (d) $\{n \in \mathbb{N} : n \text{ is even}\}$
   (e) $\{n \in \mathbb{N} : n \text{ is prime}\}$
   (f) $\{n \in \mathbb{N} : n \text{ is even and prime}\}$
   (g) $\{n \in \mathbb{N} : n \text{ is even or prime}\}$
10. How many elements are there in the following sets? Write ∞ if the set is infinite.
   (a) $\{-1, 1\}$ (b) $[-1, 1]$
   (c) $(-1, 1)$ (d) $\{n \in \mathbb{Z} : -1 \le n \le 1\}$
   (e) Σ^* where $\Sigma = \{a, b, c\}$
   (f) $\{w \in \Sigma^* : \text{length}(w) \le 4\}$ where $\Sigma = \{a, b, c\}$
11. Consider the sets
   $A = \{n \in \mathbb{P} : n \text{ is odd}\}$
   $B = \{n \in \mathbb{P} : n \text{ is prime}\}$
   $C = \{4n+3 : n \in \mathbb{P}\}$
   $D = \{x \in \mathbb{R} : x^2 - 8x + 15 = 0\}$
   Which of these sets are subsets of which? Consider all 16 possibilities.
12. Consider the sets $\{0, 1\}$, $(0, 1)$ and $[0, 1]$. True or False.
   (a) $\{0, 1\} \subseteq (0, 1)$ (b) $\{0, 1\} \subseteq [0, 1]$
   (c) $(0, 1) \subseteq [0, 1]$ (d) $\{0, 1\} \subseteq \mathbb{Z}$
   (e) $[0, 1] \subseteq \mathbb{Z}$ (f) $[0, 1] \subseteq \mathbb{Q}$
   (g) $1/2$ and $\pi/4$ are in $\{0, 1\}$
   (h) $1/2$ and $\pi/4$ are in $(0, 1)$
   (i) $1/2$ and $\pi/4$ are in $[0, 1]$
13. Consider the following three alphabets: $\Sigma_1 = \{a, b, c\}$, $\Sigma_2 = \{a, b, ca\}$, and $\Sigma_3 = \{a, b, Ab\}$. Determine to which of Σ_1^*, Σ_2^*, and Σ_3^* each word below belongs, and give its length as a member of each set to which it belongs.
   (a) *aba* (b) *bAb* (c) *cba*
   (d) *cab* (e) *caab* (f) *baAb*
14. Here is a question to think about. Let $\Sigma = \{a, b\}$ and imagine, if you can, a dictionary for all the nonempty words of Σ^* with the words arranged in the usual alphabetical order. All the words *a, aa, aaa, aaaa*, etc., must appear before the word *ba*. How far into the dictionary will you have to dig to find the word *ba*? How would the answer change if the dictionary contained only those words in Σ^* of length 5 or less?
15. Suppose that w is a nonempty word in Σ^*.

(a) If the first [i.e., leftmost] letter of w is deleted, is the resulting string in Σ^*?

(b) How about deleting letters from both ends of w? Are the resulting strings still in Σ^*?

(c) If you had a device that could recognize letters in Σ and could delete letters from strings, how could you use it to determine if an arbitrary string of symbols is in Σ^*?

Answers

1. (a) 0, 5, 10, 15, 20, say. (c) $\emptyset$, $\{1\}$, $\{2, 3\}$, $\{3, 4\}$, $\{5\}$, say.
(e) 1, 1/2, 1/3, 1/4, 1/73, say. (g) 1, 2, 4, 16, 18, say.

3. (a) λ, a, ab, cab, ba, say. (c) $aaaa$, $aaab$, $aabb$, etc.
The sets in parts (a) and (b) contain the empty word λ.

5. (a) $\emptyset$. (c) $\emptyset$. (e) $\{2, 3, 5, 7, 11, 13\}$.

7. (a) $\{0, 4, 8, 12, 16, 20\}$. (c) $\{0, 1, 2, 3, 4\}$.

9. (a) ∞. (c) ∞. (e) ∞. (g) ∞.

11. $A \subseteq A$, $B \subseteq B$, C is a subset of A, and C, D are subsets of A, B, and D.

13. (a) aba is in all three and has length 3 in each.
(c) cba is in Σ_1^* and length$(cba) = 3$.
(e) $caab$ is in Σ_1^* with length 4 and is in Σ_2^* with length 3.

15. (a) Yes.
(c) Delete first letters from the string until no longer possible. If λ is reached, the original string is in Σ^*. Otherwise, it isn't.

Set Operations

1. Let $U = \{1, 2, 3, 4, 5, \ldots, 12\}$, $A = \{1, 3, 5, 7, 9, 11\}$, $B = \{2, 3, 5, 7, 11\}$, $C = \{2, 3, 6, 12\}$, and $D = \{2, 4, 8\}$. Determine the sets
   (a) $A \cup B$ (b) $A \cap C$ (c) $(A \cup B) \cap C^c$
   (d) $A \setminus B$ (e) $C \setminus D$ (f) $B \oplus D$
   (g) How many subsets of C are there?
2. Let $A = \{1, 2, 3\}$, $B = \{n \in \mathbb{P} : n \text{ is even}\}$, and $C = \{n \in \mathbb{P} : n \text{ is odd}\}$.
   (a) Determine $A \cap B$, $B \cap C$, $B \cup C$, and $B \oplus C$.
   (b) List all subsets of A.
   (c) Which of the following sets are infinite? $A \oplus B$, $A \oplus C$, $A \setminus C$, $C \setminus A$.
3. In this exercise the universe is $\mathbb{R}$. Determine the following sets.
   (a) $[0, 3] \cap [2, 6]$ (b) $[0, 3] \cup [2, 6]$ (c) $[0, 3] \setminus [2, 6]$
   (d) $[0, 3] \oplus [2, 6]$ (e) $[0, 3]^c$ (f) $[0, 3] \cap \emptyset$
   (g) $[0, \infty) \cap \mathbb{Z}$ (h) $[0, \infty) \cap (-\infty, 2]$
   (i) $([0, \infty) \cup (-\infty, 2])^c$
4. Let $\Sigma = \{a, b\}$, $A = \{a, b, aa, bb, aaa, bbb\}$, $B = \{w \in \Sigma^* : \text{length}(w) \geq 2\}$, and $C = \{w \in \Sigma^* : \text{length}(w) \leq 2\}$.
   (a) Determine $A \cap C$, $A \setminus C$, $C \setminus A$, and $A \oplus C$.
   (b) Determine $A \cap B$, $B \cap C$, $B \cup C$, and $B \setminus A$.
   (c) Determine $\Sigma^* \setminus B$, $\Sigma \setminus B$, and $\Sigma \setminus C$.
   (d) List all subsets of Σ.
   (e) How many sets are there in $\mathcal{P}(\Sigma)$?
5. In this exercise the universe is Σ^*, where $\Sigma = \{a, b\}$. Let A, B, and C be as in Exercise 4. Determine the following sets.
   (a) $B^c \cap C^c$ (b) $(B \cap C)^c$ (c) $(B \cup C)^c$
   (d) $B^c \cup C^c$ (e) $A^c \cap C$ (f) $A^c \cap B^c$
   (g) Which of these sets are equal? Why?
6. The following statements about sets are false. For each statement, give an example, i.e., a choice of sets, for which the statement is false. Such examples are called **counterexamples**. They are examples that are counter to, i.e., contrary to, the assertion.
   (a) $A \cup B \subseteq A \cap B$ for all A, B.
   (b) $A \cap \emptyset = A$ for all A.
   (c) $A \cap (B \cup C) = (A \cap B) \cup C$ for all A, B, C.
7. For any set A, what is $A \oplus A$? $A \oplus \emptyset$?
8. For the sets $A = \{1, 3, 5, 7, 9, 11\}$ and $B = \{2, 3, 5, 7, 11\}$, determine the following numbers.
   (a) $|A|$ (b) $|B|$ (c) $|A \cup B|$
   (d) $|A| + |B| - |A \cap B|$
   (e) Do you see a general reason why the answers to (c) and (d) have to be the same?
9. The following statements about sets are false. Give a counterexample [see Exercise 6] to each statement.
   (a) $A \cap B = A \cap C$ implies $B = C$.
   (b) $A \cup B = A \cup C$ implies $B = C$.
   (c) $A \subseteq B \cup C$ implies $A \subseteq B$ or $A \subseteq C$.
10. (a) Show that relative complementation is not commutative; that is, the equality $A \setminus B = B \setminus A$ can fail.
    (b) Show that relative complementation is not associative: $(A \setminus B) \setminus C = A \setminus (B \setminus C)$ can fail.
11. Let $A = \{a, b, c\}$ and $B = \{a, b, d\}$.
    (a) List or draw the ordered pairs in $A \times A$.
    (b) List or draw the ordered pairs in $A \times B$.
    (c) List or draw the set $\{(x, y) \in A \times B : x = y\}$.
12. Let $S = \{0, 1, 2, 3, 4\}$ and $T = \{0, 2, 4\}$.
    (a) How many ordered pairs are in $S \times T$? $T \times S$?
    (b) List or draw the elements in the set $\{(m, n) \in S \times T : m < n\}$.
    (c) List or draw the elements in the set $\{(m, n) \in T \times S : m < n\}$.
    (d) List or draw the elements in the set $\{(m, n) \in S \times T : m + n \geq 3\}$.
    (e) List or draw the elements in the set $\{(m, n) \in T \times S : mn \geq 4\}$.
    (f) List or draw the elements in the set $\{(m, n) \in S \times S : m + n = 10\}$.
13. For each of the following sets, list all elements if the set has fewer than seven elements. Otherwise, list exactly seven elements of the set.
    (a) $\{(m, n) \in \mathbb{N}^2 : m = n\}$
    (b) $\{(m, n) \in \mathbb{N}^2 : m + n \text{ is prime}\}$
    (c) $\{(m, n) \in \mathbb{P}^2 : m = 6\}$
    (d) $\{(m, n) \in \mathbb{P}^2 : \min\{m, n\} = 3\}$
    (e) $\{(m, n) \in \mathbb{P}^2 : \max\{m, n\} = 3\}$
    (f) $\{(m, n) \in \mathbb{N}^2 : m^2 = n\}$
14. Draw a Venn diagram for four sets A, B, C, and D. Be sure to have a region for each of the 16 possible sets such as $A \cap B^c \cap C^c \cap D$. *Note:* This problem cannot be done using just circles, but it can be done using rectangles.

Answers

1. (a) $\{1, 2, 3, 5, 7, 9, 11\}$. (c) $\{1, 5, 7, 9, 11\}$.
(e) $\{3, 6, 12\}$. (g) 16.

3. (a) $[2, 3]$. (c) $[0, 2)$. (e) $(-\infty, 0) \cup (3, \infty)$.
(g) $\mathbb{N}$. (i) $\emptyset$.

5. (a) $\emptyset$. (c) $\emptyset$. (e) $\{\lambda, ab, ba\}$.
(g) $B^c \cap C^c$ and $(B \cup C)^c$ are equal by a De Morgan law [or by calculation], as are $(B \cap C)^c$ and $B^c \cup C^c$.

7. $A \oplus A = \emptyset$ and $A \oplus \emptyset = A$.

9. (a) Make A very small, like $A = \emptyset$.
(c) Try $A = B \cup C$ with B and C disjoint.

11. (a) (a, a), (a, b), (a, c), (b, a), etc. There are nine altogether.
(c) (a, a), (b, b).

13. (a) $(0, 0)$, $(1, 1)$, $(2, 2)$, $\ldots$, $(6, 6)$, say.
(c) $(6, 1)$, $(6, 2)$, $(6, 3)$, $\ldots$, $(6, 7)$, say.
(e) $(1, 3)$, $(2, 3)$, $(3, 3)$, $(3, 2)$, $(3, 1)$.

Functions

1. Let $f(n) = n^2 + 3$ and $g(n) = 5n - 11$ for $n \in \mathbb{N}$. Thus $f: \mathbb{N} \to \mathbb{N}$ and $g: \mathbb{N} \to \mathbb{Z}$. Calculate

   (a) $f(1)$ and $g(1)$ (b) $f(2)$ and $g(2)$

   (c) $f(3)$ and $g(3)$ (d) $f(4)$ and $g(4)$

   (e) $f(5)$ and $g(5)$

   (f) To think about: Is $f(n) + g(n)$ always an even number?

2. Consider the function $h: \mathbb{P} \to \mathbb{P}$ defined by $h(n) = |\{k \in \mathbb{N} : k|n\}|$ for $n \in \mathbb{P}$. In words, $h(n)$ is the number of divisors of n. Calculate $h(n)$ for $1 \le n \le 10$ and for $n = 73$.

3. Let Σ^* be the language using letters from $\Sigma = \{a, b\}$. We've already seen a useful function from Σ^* to $\mathbb{N}$. It is the length function, which already has a name: length. Calculate

   (a) length(bab) (b) length($aaaaaaa$)

   (c) length(λ)

   (d) What is the image set Im(length) for this function? Explain.

4. The codomain of a function doesn't have to consist of numbers either. Let Σ^* be as in Exercise 3, and define

$$g(n) = \{w \in \Sigma^* : \text{length}(w) \le n\} \quad \text{for} \quad n \in \mathbb{N}.$$

   Thus $g: \mathbb{N} \to \mathcal{P}(\Sigma^*)$. Determine

   (a) $g(0)$ (b) $g(1)$ (c) $g(2)$

   (d) Are all the sets $g(n)$ finite?

   (e) Give an example of a set in $\mathcal{P}(\Sigma^*)$ that is not in the image set Im(g).

5. Let f be the function in Example 3.

   (a) Calculate $f(0,0)$, $f(8,8)$, $f(-8,-8)$, $f(73,73)$, and $f(-73,-73)$.

   (b) Find $f(n,n)$ for all (n,n) in $\mathbb{Z} \times \mathbb{Z}$. *Hint:* Consider the cases when n is even and when it is odd.

6. The greatest common divisor gcd defines a function on the product set $\mathbb{P} \times \mathbb{P}$. It already has a fine name: gcd.

   (a) Calculate gcd(7, 14), gcd(14, 28), and gcd(1001, 2002).

   (b) What is gcd(n, $2n$) for all $n \in \mathbb{P}$?

   (c) What is the image set Im(gcd)?

7. We define $f: \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x^3 & \text{if } x \ge 1, \\ x & \text{if } 0 \le x < 1, \\ -x^3 & \text{if } x < 0. \end{cases}$$

   (a) Calculate $f(3)$, $f(1/3)$, $f(-1/3)$, and $f(-3)$.

   (b) Sketch a graph of f.

   (c) Find Im(f).

8. Let $S = \{1, 2, 3, 4, 5\}$ and consider the functions 1_S, f, g, and h from S into S defined by $1_S(n) = n$, $f(n) = 6 - n$, $g(n) = \max\{3, n\}$, and $h(n) = \max\{1, n-1\}$.

   (a) Write each of these functions as a set of ordered pairs, i.e., list the elements in their graphs.

   (b) Sketch a graph of each of these functions.

9. For $n \in \mathbb{Z}$, let $f(n) = \frac{1}{2}[(-1)^n + 1]$. The function f is the characteristic function for some subset of $\mathbb{Z}$. Which subset?

10. Consider subsets A and B of a set S.

   (a) The function $\chi_A \cdot \chi_B$ is the characteristic function of some subset of S. Which subset? Explain.

   (b) Repeat part (a) for the function $\chi_A + \chi_B - \chi_{A \cap B}$.

   (c) Repeat part (a) for the function $\chi_A + \chi_B - 2 \cdot \chi_{A \cap B}$.

11. Here we consider two functions that are defined in terms of the floor and ceiling functions.

   (a) Let $f(n) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{3} \rfloor$ for $n \in \mathbb{N}$. Calculate $f(n)$ for $0 \le n \le 10$ and for $n = 73$.

   (b) Let $g(n) = \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor$ for $n \in \mathbb{Z}$. g is the characteristic function of some subset of $\mathbb{Z}$. What is the subset?

12. In Example 5(b), we compared the functions $\sqrt{\log x}$ and $\log \sqrt{x}$. Show that these functions take the same value for $x = 10{,}000$.

13. We define functions mapping $\mathbb{R}$ into $\mathbb{R}$ as follows: $f(x) = x^3 - 4x$, $g(x) = 1/(x^2 + 1)$, $h(x) = x^4$. Find

   (a) $f \circ f$ (b) $g \circ g$ (c) $h \circ g$

   (d) $g \circ h$ (e) $f \circ g \circ h$ (f) $f \circ h \circ g$

   (g) $h \circ g \circ f$

14. Repeat Exercise 13 for the functions $f(x) = x^2$, $g(x) = \sqrt{x^2 + 1}$, and $h(x) = 3x - 1$.

15. Consider the functions f and g mapping $\mathbb{Z}$ into $\mathbb{Z}$, where $f(n) = n - 1$ for $n \in \mathbb{Z}$ and g is the characteristic function χ_E of $E = \{n \in \mathbb{Z} : n \text{ is even}\}$.

   (a) Calculate $(g \circ f)(5)$, $(g \circ f)(4)$, $(f \circ g)(7)$, and $(f \circ g)(8)$.

   (b) Calculate $(f \circ f)(11)$, $(f \circ f)(12)$, $(g \circ g)(11)$, and $(g \circ g)(12)$.

   (c) Determine the functions $g \circ f$ and $f \circ f$.

   (d) Show that $g \circ g = g \circ f$ and that $f \circ g$ is the negative of $g \circ f$.

16. Several important functions can be found on hand-held calculators. Why isn't the identity function, i.e., the function $1_{\mathbb{R}}$, where $1_{\mathbb{R}}(x) = x$ for all $x \in \mathbb{R}$, among them?

Answers

1. (a) 4, −6. (c) 12, 4. (e) 28, 14.

3. (a) 3. (c) 0.

5. (a) 1, 1, 1, 0, 0. See the answer to part (b).
(b) $f(n,n) = 1$ for even n, and $f(n,n) = 0$ for odd n. This can be checked by calculation or by applying the theorem on page 5, with $k = 2$.

7. (a) $f(3) = 27$, $f(1/3) = 1/3$, $f(-1/3) = 1/27$, $f(-3) = 27$.
(c) $\text{Im}(f) = [0, \infty)$.

9. $\{n \in \mathbb{Z} : n \text{ is even}\}$.

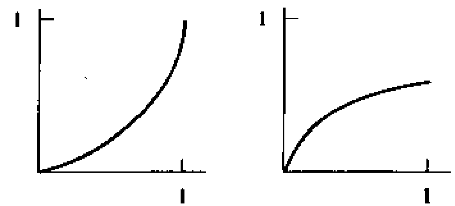
11. (a) The answers for $n = 0, 1, 2, 3, 4, 5$ are 0, 0, 1, 2, 3, 3. The remaining answers are 5, 5, 6, 7, 8, 60.

13. (a) $f \circ f(x) = (x^3 - 4x)^3 - 4(x^3 - 4x)$.
(c) $h \circ g(x) = (x^2 + 1)^{-4}$.
(e) $f \circ g \circ h(x) = (x^8 + 1)^{-3} - 4(x^8 + 1)^{-1}$.
(g) $h \circ g \circ f(x) = [(x^3 - 4x)^2 + 1]^{-4}$.

15. (a) 1, 0, −1, and 0.
(c) $g \circ f$ is the characteristic function of $\mathbb{Z} \setminus E$. $f \circ f(n) = n - 2$ for all $n \in \mathbb{Z}$.

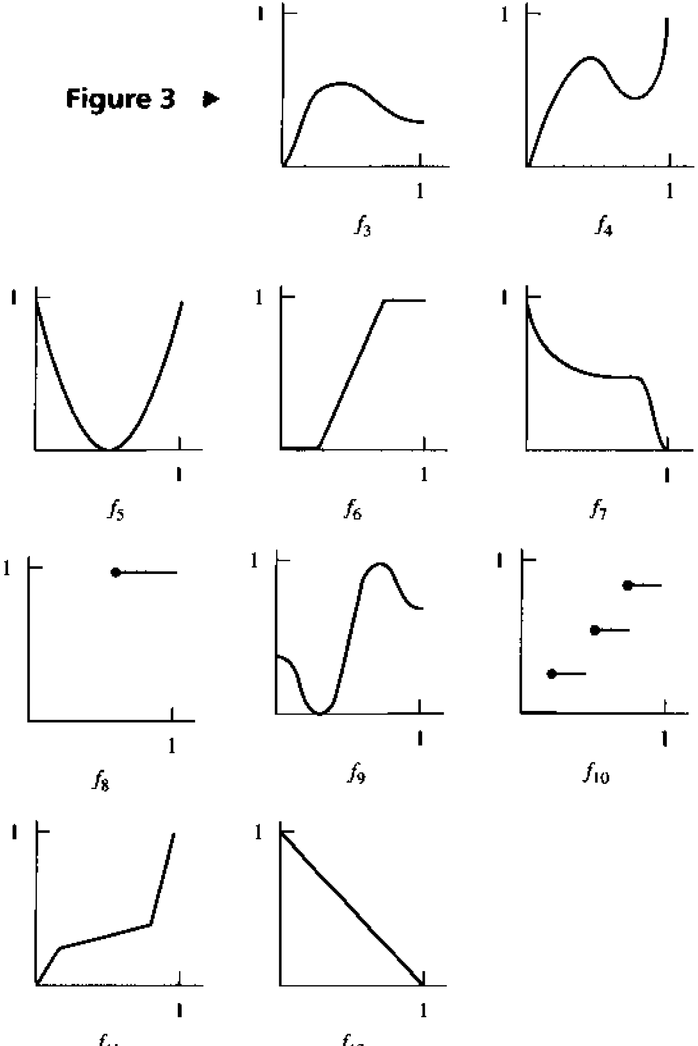
Properties of Functions

1. Let $S = \{1, 2, 3, 4, 5\}$ and $T = \{a, b, c, d\}$. For each question below: if the answer is YES, give an example; if the answer if NO, explain briefly.
   (a) Are there any one-to-one functions from S into T?
   (b) Are there any one-to-one functions from T into S?
   (c) Are there any functions mapping S onto T?
   (d) Are there any functions mapping T onto S?
   (e) Are there any one-to-one correspondences between S and T?
2. The functions sketched in Figure 3 have domain and codomain both equal to $[0,1]$.
   (a) Which of these functions are one-to-one?
   (b) Which of these functions map $[0, 1]$ onto $[0, 1]$?
   (c) Which of these functions are one-to-one correspondences?
3. The function $f(m, n) = 2^m 3^n$ is a one-to-one function from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$.
   (a) Calculate $f(m, n)$ for five different elements (m, n) in $\mathbb{N} \times \mathbb{N}$.
   (b) Explain why f is one-to-one.
   (c) Does f map $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$? Explain.
   (d) Show that $g(m, n) = 2^m 4^n$ defines a function on $\mathbb{N} \times \mathbb{N}$ that is not one-to-one.
4. Consider the following functions from $\mathbb{N}$ into $\mathbb{N}$: $1_{\mathbb{N}}(n) = n$, $f(n) = 3n$, $g(n) = n + (-1)^n$, $h(n) = \min\{n, 100\}$, $k(n) = \max\{0, n - 5\}$.
   (a) Which of these functions are one-to-one?
   (b) Which of these functions map $\mathbb{N}$ onto $\mathbb{N}$?
5. Here are two "shift functions" mapping $\mathbb{N}$ into $\mathbb{N}$: $f(n) = n + 1$ and $g(n) = \max\{0, n - 1\}$ for $n \in \mathbb{N}$.
   (a) Calculate $f(n)$ for $n = 0, 1, 2, 3, 4, 73$.
   (b) Calculate $g(n)$ for $n = 0, 1, 2, 3, 4, 73$.
   (c) Show that f is one-to-one but does not map $\mathbb{N}$ onto $\mathbb{N}$.
   (d) Show that g maps $\mathbb{N}$ onto $\mathbb{N}$ but is not one-to-one.
   (e) Show that $g \circ f(n) = n$ for all n, but that $f \circ g(n) = n$ does not hold for all n.
6. Let $\Sigma = \{a, b, c\}$ and let Σ^* be the set of all words w using letters from Σ; see Example 2(b). Define $L(w) = \text{length}(w)$ for all $w \in \Sigma^*$.
   (a) Calculate $L(w)$ for the words $w_1 = cab$, $w_2 = ababac$, and $w_3 = \lambda$.
   (b) Is L a one-to-one function? Explain.
   (c) The function L maps Σ^* into $\mathbb{N}$. Does L map Σ^* onto $\mathbb{N}$? Explain.
   (d) Find all words w such that $L(w) = 2$.
7. Find the inverses of the following functions mapping $\mathbb{R}$ into $\mathbb{R}$.
   (a) $f(x) = 2x + 3$
   (b) $g(x) = x^3 - 2$
   (c) $h(x) = (x - 2)^3$
   (d) $k(x) = \sqrt[3]{x} + 7$
8. Many hand-held calculators have the functions $\log x$, x^2, $\sqrt{x}$, and $1/x$.
   (a) Specify the domains of these functions.
   (b) Which of these functions are inverses of each other?
   (c) Which pairs of these functions commute with respect to composition?
   (d) Some hand-held calculators also have the functions $\sin x$, $\cos x$, and $\tan x$. If you know a little trigonometry, repeat parts (a), (b), and (c) for these functions.
9. Show that the following functions are their own inverses.
   (a) The function $f: (0, \infty) \to (0, \infty)$ given by $f(x) = 1/x$.
   (b) The function $\phi: \mathcal{P}(S) \to \mathcal{P}(S)$ defined by $\phi(A) = A^c *$.
   (c) The function $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = 1 - x$.
10. Let A be a subset of some set S and consider the characteristic function χ_A of A. Find $\chi_A^{\leftarrow}(1)$ and $\chi_A^{\leftarrow}(0)$.
11. Here are some functions from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$: $\text{SUM}(m, n) = m + n$, $\text{PROD}(m, n) = m * n$, $\text{MAX}(m, n) = \max\{m, n\}$, $\text{MIN}(m, n) = \min\{m, n\}$; here $*$ denotes multiplication of integers.
    (a) Which of these functions map $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$?
    (b) Show that none of these functions are one-to-one.
    (c) For each of these functions F, how big is the set $F^{\leftarrow}(4)$?
12. Consider the function $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by
$$f(x, y) = (x + y, x - y).$$
This function is invertible. Show that the inverse function is given by
$$f^{-1}(a, b) = \left(\frac{a + b}{2}, \frac{a - b}{2}\right)$$
for all (a, b) in $\mathbb{R} \times \mathbb{R}$.
13. Let $f: S \to T$ and $g: T \to U$ be one-to-one functions. Show that the function $g \circ f: S \to U$ is one-to-one.
14. Let $f: S \to T$ be an invertible function. Show that f^{-1} is invertible and that $(f^{-1})^{-1} = f$.
15. Let $f: S \to T$ and $g: T \to U$ be invertible functions. Show that $g \circ f$ is invertible and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.



f_1 f_2

Figure 3 ▶



Answers

1. (a) No; S is bigger than T.

(c) Yes. For example, let $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = f(5) = d$.

(e) No. This follows from either part (a) or part (d).

3. (a) $f(2, 1) = 2^2 3^1 = 12$, $f(1, 2) = 2^1 3^2 = 18$, etc.

(c) Consider 5, for instance. 5 is not in $\text{Im}(f)$.

5. (a) $f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 5$, $f(73) = 74$.

(c) For one-to-oneness, observe that if $f(n) = f(n')$,

not map onto $\mathbb{N}$ because $0 \notin \text{Im}(f)$.

(e) $g(f(n)) = \max\{0, (n + 1) - 1\} = n$, but $f(g(0)) = f(0) = 1$.

7. (a) $f^{-1}(y) = (y - 3)/2$. (c) $h^{-1}(y) = 2 + \sqrt[3]{y}$.

9. (a) $(f \circ f)(x) = 1/(1/x) = x$.

(b) and (c) are similar verifications.

11. (a) All of them; verify this.

(c) $\text{SUM}^{\leftarrow}(4)$ has 5 elements, $\text{PROD}^{\leftarrow}(4)$ has 3 elements, $\text{MAX}^{\leftarrow}(4)$ has 9 elements, and $\text{MIN}^{\leftarrow}(4)$ is infinite.

13. If $s_1 \neq s_2$, then $f(s_1) \neq f(s_2)$ [why?]. Thus $g(f(s_1)) \neq g(f(s_2))$ [why?]. Hence $g \circ f$ is one-to-one.

15. Since f and g are invertible, the functions $f^{-1}: T \to S$, $g^{-1}: U \to T$, and $f^{-1} \circ g^{-1}: U \to S$ exist. So it suffices to show $(g \circ f) \circ (f^{-1} \circ g^{-1}) = 1_U$ and $(f^{-1} \circ g^{-1}) \circ (g \circ f) = 1_S$. One can show directly that $g \circ f$ is one-to-one [see Exercise 13] and onto, but it is actually easier—and more useful—to verify that $f^{-1} \circ g^{-1}$ has the properties of the inverse to $g \circ f$.

1. For the following relations on $S = \{0, 1, 2, 3\}$, specify which of the properties (R), (AR), (S), (AS), and (T) the relations satisfy.
   (a) $(m, n) \in R_1$ if $m + n = 3$
   (b) $(m, n) \in R_2$ if $m - n$ is even
   (c) $(m, n) \in R_3$ if $m \leq n$
   (d) $(m, n) \in R_4$ if $m + n \leq 4$
   (e) $(m, n) \in R_5$ if $\max\{m, n\} = 3$
2. Let $A = \{0, 1, 2\}$. Each of the statements below defines a relation R on A by $(m, n) \in R$ if the statement is true for m and n. Write each of the relations as a set of ordered pairs.
   (a) $m \leq n$
   (b) $m < n$
   (c) $m = n$
   (d) $mn = 0$
   (e) $mn = m$
   (f) $m + n \in A$
   (g) $m^2 + n^2 = 2$
   (h) $m^2 + n^2 = 3$
   (i) $m = \max\{n, 1\}$
3. Which of the relations in Exercise 2 are reflexive? symmetric?
4. The following relations are defined on $\mathbb{N}$.
   (a) Write the relation R_1 defined by $(m, n) \in R_1$ if $m + n = 5$ as a set of ordered pairs.
   (b) Do the same for R_2 defined by $\max\{m, n\} = 2$.
   (c) The relation R_3 defined by $\min\{m, n\} = 2$ consists of infinitely many ordered pairs. List five of them.
5. For each of the relations in Exercise 4, specify which of the properties (R), (AR), (S), (AS), and (T) the relation satisfies.
6. Consider the relation R on $\mathbb{Z}$ defined by $(m, n) \in R$ if and only if $m^3 - n^3 \equiv 0 \pmod 5$. Which of the properties (R), (AR), (S), (AS), and (T) are satisfied by R?
7. Define the "divides" relation R on $\mathbb{N}$ by

$$(m, n) \in R \quad \text{if} \quad m|n.$$

   [Recall from §1.2 that $m|n$ means that n is a multiple of m.]
   (a) Which of the properties (R), (AR), (S), (AS), and (T) does R satisfy?
   (b) Describe the converse relation $R^{\leftarrow}$.
   (c) Which of the properties (R), (AR), (S), (AS), and (T) does the converse relation $R^{\leftarrow}$ satisfy?
8. What is the connection between a relation R and the relation $(R^{\leftarrow})^{\leftarrow}$?
9. (a) If S is a nonempty set, then the empty set $\emptyset$ is a subset of $S \times S$, so it is a relation on S, called the the **empty relation**. Which of the properties (R), (AR), (S), (AS), and (T) does $\emptyset$ possess?
   (b) Repeat part (a) for the **universal relation** $U = S \times S$ on S.
10. Give an example of a relation that is:
    (a) antisymmetric and transitive but not reflexive,
    (b) symmetric but not reflexive or transitive.
11. Do a relation and its converse always satisfy the same conditions (R), (AR), (S), and (AS)? Explain.
12. Show that a relation R is transitive if and only if its converse relation $R^{\leftarrow}$ is transitive.
13. Let R_1 and R_2 be relations on a set S.
    (a) Show that $R_1 \cap R_2$ is reflexive if R_1 and R_2 are.
    (b) Show that $R_1 \cap R_2$ is symmetric if R_1 and R_2 are.
    (c) Show that $R_1 \cap R_2$ is transitive if R_1 and R_2 are.
14. Let R_1 and R_2 be relations on a set S.
    (a) Must $R_1 \cup R_2$ be reflexive if R_1 and R_2 are?
    (b) Must $R_1 \cup R_2$ be symmetric if R_1 and R_2 are?
    (c) Must $R_1 \cup R_2$ be transitive if R_1 and R_2 are?
15. Let R be a relation on a set S.
    (a) Prove that R is symmetric if and only if $R = R^{\leftarrow}$.
    (b) Prove that R is antisymmetric if and only if $R \cap R^{\leftarrow} \subseteq E$, where $E = \{(x, x) : x \in S\}$.
16. Let R_1 and R_2 be relations from a set S to a set T.
    (a) Show that $(R_1 \cup R_2)^{\leftarrow} = R_1^{\leftarrow} \cup R_2^{\leftarrow}$.
    (b) Show that $(R_1 \cap R_2)^{\leftarrow} = R_1^{\leftarrow} \cap R_2^{\leftarrow}$.
    (c) Show that if $R_1 \subseteq R_2$ then $R_1^{\leftarrow} \subseteq R_2^{\leftarrow}$.
17. Draw pictures of each of the relations in Exercise 1. Don't use arrows if the relation is symmetric.
18. Draw pictures of each of the relations in Exercise 2. Don't use arrows if the relation is symmetric.

Answers

1. (a) R_1 satisfies (AR) and (S).
(c) R_3 satisfies (R), (AS), and (T).
(e) R_5 satisfies only (S).

3. The relations in (a) and (c) are reflexive. The relations in (c), (d), (f), (g), and (h) are symmetric.

5. R_1 satisfies (AR) and (S). R_2 and R_3 satisfy only (S).

7. (a) The divides relation satisfies (R), (AS), and (T).
(c) The converse relation $R^{\leftarrow}$ also satisfies (R), (AS), and (T).

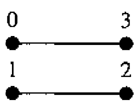
9. (a) The empty relation satisfies (AR), (S), (AS), and (T). The last three properties hold vacuously.

11. Yes. For (R) and (AR), observe that $(x, x) \in R \iff (x, x) \in R^{\leftarrow}$. For (S) and (AS), just interchange x and y in the conditions for R to get the conditions for $R^{\leftarrow}$. There is no change in meaning.

13. (a) If $E \subseteq R_1$ and $E \subseteq R_2$, then $E \subseteq R_1 \cap R_2$. Alternatively, if R_1 and R_2 are reflexive and $x \in S$, then $(x, x) \in R_1$ and $(x, x) \in R_2$; so $(x, x) \in R_1 \cap R_2$.
(c) Suppose R_1 and R_2 are transitive. If $(x, y), (y, z) \in R_1 \cap R_2$, then $(x, y), (y, z) \in R_1$, so $(x, z) \in R_1$. Similarly, $(x, z) \in R_2$.

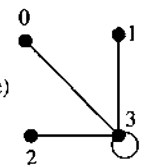
15. (a) Suppose R is symmetric. If $(x, y) \in R$, then $(y, x) \in R$ by symmetry, so $(x, y) \in R^{\leftarrow}$. Similarly, $(x, y) \in R^{\leftarrow}$ implies $(x, y) \in R$ [check] so that $R = R^{\leftarrow}$. For the converse, suppose that $R = R^{\leftarrow}$ and show R is symmetric.

17. (a)



(c) See Figure 1(a).

(e)



Relations

1. Which of the following describe equivalence relations? For those that are not equivalence relations, specify which of (R), (S), and (T) fail, and illustrate the failures with examples.

   (a) $L_1 \| L_2$ for straight lines in the plane if L_1 and L_2 are the same or are parallel.

   (b) $L_1 \perp L_2$ for straight lines in the plane if L_1 and L_2 are perpendicular.

   (c) $p_1 \sim p_2$ for Americans if p_1 and p_2 live in the same state.

   (d) $p_1 \approx p_2$ for Americans if p_1 and p_2 live in the same state or in neighboring states.

   (e) $p_1 \approx p_2$ for people if p_1 and p_2 have a parent in common.

   (f) $p_1 \cong p_2$ for people if p_1 and p_2 have the same mother.

2. For each example of an equivalence relation in Exercise 1, describe the members of some equivalence class.

3. Let S be a set. Is equality, i.e., "=", an equivalence relation?

4. Define the relation $\equiv$ on $\mathbb{Z}$ by $m \equiv n$ in case $m - n$ is even. Is $\equiv$ an equivalence relation? Explain.

5. If G and H are both graphs with vertex set $\{1, 2, \dots, n\}$, we say that G is **isomorphic** to H, and write $G \simeq H$, in case there is a way to label the vertices of G so that it becomes H. For example, the graphs in Figure 3, with vertex set $\{1, 2, 3\}$, are isomorphic by relabeling $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$.

   (a) Give a picture of another graph isomorphic to these two.

   (b) Find a graph with vertex set $\{1, 2, 3\}$ that is not isomorphic to the graphs in Figure 3, yet has three edges, exactly one of which is a loop.

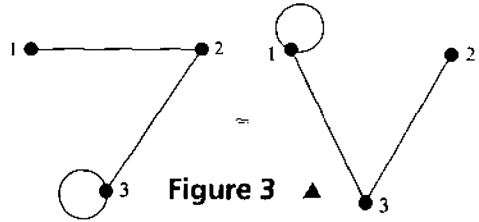


Figure 3 ▲

   (c) Find another example as in part (b) that isn't isomorphic to the one you found in part (b) [or the ones in Figure 3].

   (d) Show that $\simeq$ is an equivalence relation on the set of all graphs with vertex set $\{1, 2, \dots, n\}$.

6. Can you think of situations in life where you'd use the term "equivalent" and where a natural equivalence relation is involved?

7. Define the relation $\approx$ on $\mathbb{Z}$ by $m \approx n$ in case $m^2 = n^2$.

   (a) Show that $\approx$ is an equivalence relation on $\mathbb{Z}$.

   (b) Describe the equivalence classes for $\approx$. How many are there?

8. (a) For $m, n \in \mathbb{Z}$, define $m \sim n$ in case $m - n$ is odd. Is the relation $\sim$ reflexive? symmetric? transitive? Is $\sim$ an equivalence relation?

   (b) For a and b in $\mathbb{R}$, define $a \sim b$ in case $|a - b| \le 1$. One could say that $a \sim b$ in case a and b are "close enough" or "approximately equal." Answer the questions in part (a).

9. Consider the functions g and h mapping $\mathbb{Z}$ into $\mathbb{N}$ defined by $g(n) = |n|$ and $h(n) = 1 + (-1)^n$.

   (a) Describe the sets in the partition $\{g^{\leftarrow}(k) : k \text{ is in the codomain of } g\}$ of $\mathbb{Z}$. How many sets are there?

   (b) Describe the sets in the partition $\{h^{\leftarrow}(k) : k \text{ is in the codomain of } h\}$ of $\mathbb{Z}$. How many sets are there?

10. On the set $\mathbb{N} \times \mathbb{N}$ define $(m, n) \sim (k, l)$ if $m + l = n + k$.

    (a) Show that $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

    (b) Draw a sketch of $\mathbb{N} \times \mathbb{N}$ that shows several equivalence classes.

11. Let Σ be an alphabet, and for w_1 and w_2 in Σ^* define $w_1 \sim w_2$ if $\text{length}(w_1) = \text{length}(w_2)$. Explain why $\sim$ is an equivalence relation, and describe the equivalence classes.

12. Let P be a set of computer programs, and regard programs p_1 and p_2 as equivalent if they always produce the same outputs for given inputs. Is this an equivalence relation on P? Explain.

13. Consider $\mathbb{Z} \times \mathbb{P}$ and define $(m, n) \sim (p, q)$ if $mq = np$.

    (a) Show that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{P}$.

    (b) Show that $\sim$ is the equivalence relation corresponding to the function $\mathbb{Z} \times \mathbb{P} \to \mathbb{Q}$ given by $f(m, n) = \frac{m}{n}$; see Theorem 2(a).

14. In the proof of Theorem 2(b), we obtained the equality $\nu^{\leftarrow}([s]) = [s]$. Does this mean that the function ν has an inverse and that the inverse of ν is the identity function on $[S]$? Discuss.

15. As in Exercise 7 define $\approx$ on $\mathbb{Z}$ by $m \approx n$ in case $m^2 = n^2$.

    (a) What is wrong with the following "definition" of $\le$ on $[\mathbb{Z}]$? Let $[m] \le [n]$ if and only if $m \le n$.

    (b) What, if anything, is wrong with the following "definition" of a function $f : [\mathbb{Z}] \to \mathbb{Z}$? Let $f([m]) = m^2 + m + 1$.

    (c) Repeat part (b) with $g([m]) = m^4 + m^2 + 1$.

    (d) What, if anything, is wrong with the following "definition" of the operation $\oplus$ on $[\mathbb{Z}]$? Let $[m] \oplus [n] = [m + n]$.

16. Let $\mathbb{Q}^+ = \left\{\frac{m}{n} : m, n \in \mathbb{P}\right\}$. Which of the following are well-defined definitions of functions on $\mathbb{Q}^+$?

    (a) $f\left(\frac{m}{n}\right) = \frac{n}{m}$ (b) $g\left(\frac{m}{n}\right) = m^2 + n^2$

    (c) $h\left(\frac{m}{n}\right) = \frac{m^2 + n^2}{mn}$

17. (a) Verify that the relation $\cong$ defined in Example 5(b) is an equivalence relation on $V(G)$.

(b) Given a vertex v in $V(G)$, describe in words the equivalence class containing v.

18. Let S be the set of all sequences (s_n) of real numbers, and define $(s_n) \approx (t_n)$ if $\{n \in \mathbb{N} : s_n \neq t_n\}$ is finite. Show that $\approx$ is an equivalence relation on S.

19. Show that the function θ in Example 12 is a one-to-one correspondence between the set $[S]$ of equivalence classes and the set $f(S)$ of values of f.

Answers

1. (a) is an equivalence relation.

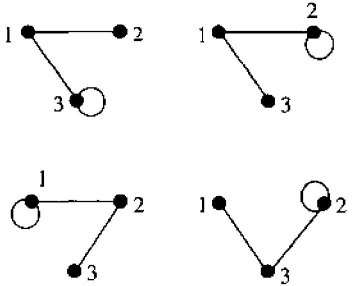
(c) There are lots of Americans who live in no state, e.g., the residents of Washington, D.C., so

(R) fails for $\sim$.

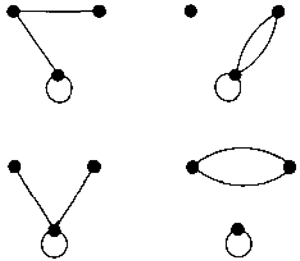
(e) is not an equivalence relation because $\approx$ is not transitive.

3. Very much so. See Example 5 on page 97.

5. (a) The possibilities are



(c) The four equivalence classes have representatives



7. (a) Verify directly, or apply Theorem 2(a) on page 117 with $f(m) = m^2$ for $m \in \mathbb{Z}$.

9. (a) There are infinitely many classes: $\{0\}$ and the classes $\{n, -n\}$ for $n \in \mathbb{P}$.

11. Apply Theorem 2, using the length function. The equivalence classes are the sets Σ^k, $k \in \mathbb{N}$.

13. (a) Use brute force or Theorem 2(a) with part (b).

15. (a) Not well-defined: depends on the representative. For example, $[3] = [-3]$ and $-3 \leq 2$. If the definition made sense, we would have $[3] = [-3] \leq [2]$ and hence $3 \leq 2$.

(c) Nothing wrong. If $[m] = [n]$, then $m^4 + m^2 + 1 = n^4 + n^2 + 1$.

17. (a) $\cong$ is reflexive by its definition, and it's symmetric since equality "=" and R are. For transitivity, consider $u \cong v$ and $v \cong w$. If $u = v$ or if $v = w$, then $u \cong w$ is clear. Otherwise, (u, v) and (v, w) are in R, so (u, w) is in R. Either way, $u \cong w$. Thus $\cong$ is transitive.

19. For one-to-one, observe that $\theta([s]) = \theta([t])$ implies $f(s) = f(t)$ implies $s \sim t$, and this implies $[s] = [t]$. Clearly, θ maps $[S]$ into $f(S)$. To see that θ maps onto $f(S)$, consider $y \in f(S)$. Then $y = f(s_0)$ for some $s_0 \in S$. Hence $[s_0]$ belongs to $[S]$ and $\theta([s_0]) = f(s_0) = y$. That is, y is in $\mathrm{Im}(\theta)$. We've shown $f(S) \subseteq \mathrm{Im}(\theta)$, so θ maps $[S]$ onto $f(S)$.

1. Explain why $n^5 - n$ is a multiple of 10 for all n in $\mathbb{P}$. *Hint:* Most of the work was done in Example 1.
2. Write a loop in the style of Figure 2 that corresponds to the proof that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ in Example 2(b).
3. (a) Show that "$n^5 - n + 1$ is a multiple of 5" is an invariant of the loop in Figure 1.

   (b) Is $n^5 - n + 1$ a multiple of 5 for all n in $\mathbb{P}$ with $n \le 37^{100}$?
4. (a) Show that $n^3 - n$ is a multiple of 6 for all n in $\mathbb{P}$.

   (b) Use part (a) to give another proof of Example 2(d).
5. Prove

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2$$
$$= \frac{n(n+1)(2n+1)}{6} \quad \text{for} \quad n \in \mathbb{P}.$$

6. Prove

$$4 + 10 + 16 + \cdots + (6n - 2) = n(3n + 1) \text{ for all } n \in \mathbb{P}.$$

7. Show each of the following.

   (a) $37^{100} - 37^{20}$ is a multiple of 10.

   (b) $37^{20} - 37^4$ is a multiple of 10.

   (c) $37^{500} - 37^4$ is a multiple of 10.

   (d) $37^4 - 1$ is a multiple of 10.

   (e) $37^{500} - 1$ is a multiple of 10.
8. Prove

$$\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1} \quad \text{for} \quad n \in \mathbb{P}.$$

9. Show by induction that, if $s_0 = a$ and $s_n = 2s_{n-1} + b$ for $n \in \mathbb{P}$, then $s_n = 2^n a + (2^n - 1)b$ for every $n \in \mathbb{N}$.
10. Consider the following procedure.

```
begin
S := 1
while 1 ≤ S do
   print S
   S := S + 2√S + 1
```

   (a) List the first four printed values of S.

   (b) Use mathematical induction to show that the value of S is always an integer. [It is easier to prove the stronger statement that the value of S is always the square of an integer; in fact, $S = n^2$ at the start of the nth pass through the loop.]
11. Prove that $11^n - 4^n$ is divisible by 7 for all n in $\mathbb{P}$.
12. (a) Choose m and $p(k)$ in the segment

```
k := m
while m ≤ k do
   if p(k) is true then
      k := k + 1
```

   so that proving $p(k)$ an invariant of the loop would show that $2^n < n!$ for all integers $n \ge 4$.

   (b) Verify that your $p(k)$ in part (a) is an invariant of the loop.

   (c) The proposition $p(k) =$ "$8^k < k!$" is an invariant of this loop. Does it follow that $8^n < n!$ for all $n \ge 4$? Explain.
13. (a) Show that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ is an invariant of the loop in the algorithm

```
begin
k := 0
while 0 ≤ k do
   k := k + 1
end
```

   (b) Repeat part (a) for the invariant $\sum_{i=0}^{k} 2^i = 2^{k+1}$.

   (c) Can you use part (a) to prove that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ for every k in $\mathbb{N}$? Explain.

   (d) Can you use part (b) to prove that $\sum_{i=0}^{k} 2^i = 2^{k+1}$ for every k in $\mathbb{N}$? Explain.
14. Prove that $n^2 > n + 1$ for $n \ge 2$.
15. (a) Calculate $1 + 3 + \cdots + (2n - 1)$ for a few values of n, and then guess a general formula for this sum.

   (b) Prove the formula obtained in part (a) by induction.
16. For which n in $\mathbb{P}$ does the inequality $4n \le n^2 - 7$ hold? Explain.
17. Consider the proposition $p(n) =$ "$n^2 + 5n + 1$ is even."

   (a) Prove that $p(k) \Longrightarrow p(k + 1)$ for all k in $\mathbb{P}$.

   (b) For which values of n is $p(n)$ actually true? What is the moral of this exercise?
18. Prove $(2n + 1) + (2n + 3) + (2n + 5) + \cdots + (4n - 1) = 3n^2$ for all n in $\mathbb{P}$. The sum can also be written $\sum_{i=n}^{2n-1} (2i + 1)$.
19. Prove that $5^n - 4n - 1$ is divisible by 16 for n in $\mathbb{P}$.

20. Prove $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$, i.e., $\sum_{i=1}^{n} i^3 = \left[\sum_{i=1}^{n} i\right]^2$ for all n in $\mathbb{P}$. *Hint:* Use the identity in Example 2(b).

21. Prove that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

for n in $\mathbb{P}$. For $n = 1$ this equation says that $\frac{1}{2} = 1 - \frac{1}{2}$, and for $n = 2$ it says that $\frac{1}{3} + \frac{1}{4} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$.

22. For n in $\mathbb{P}$, prove

(a) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$ (b) $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} - 1$

23. Prove that $5^{n+1} + 2 \cdot 3^n + 1$ is divisible by 8 for $n \in \mathbb{N}$.

24. Prove that $8^{n+2} + 9^{2n+1}$ is divisible by 73 for $n \in \mathbb{N}$.

25. This exercise requires a little knowledge of trigonometric identities. Prove that $|\sin nx| \leq n|\sin x|$ for all x in $\mathbb{R}$ and all n in $\mathbb{P}$.

Answers

Induction proofs should be written carefully and completely. These answers will serve only as guides, *not* as models.

1. This is clear, because both n^5 and n are even if n is even, and both are odd if n is odd.

3. (a) If $k^5 - k + 1$ is a multiple of 5 for some $k \in \mathbb{P}$, then, just as in Example 1,

$$(k+1)^5 - (k+1) + 1 = (k^5 - k + 1) + 5(k^4 + 2k^3 + 2k^2 + k),$$

so $(k+1)^5 - (k+1) + 1$ is also a multiple of 5.

5. Check the basis. For the inductive step, assume the equality holds for k. Then

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2.$$

Some algebra shows that the right-hand side equals

$$\frac{(k+1)(k+2)(2k+3)}{6},$$

so the equality holds for $k+1$ whenever it holds for k.

7. (a) Take $n = 37^{20}$ in Exercise 1.
(c) By (a), (b), and Exercise 1, $(37^{500} - 37^{100}) + (37^{100} - 37^{20}) + (37^{20} - 37^4)$ is a multiple of 10.
(e) By (c) and (d), as in (c).

9. The basis is "$s_0 = 2^0 a + (2^0 - 1)b$," which is true since $2^0 = 1$ and $s_0 = a$. Assume inductively that $s_k = 2^k a + (2^k - 1)b$ for some $k \in \mathbb{N}$. The algebra in the inductive step is

$$2 \cdot [2^k a + (2^k - 1)b] + b = 2^{k+1} a + 2^{k+1} b - 2b + b.$$

11. Show that $11^{k+1} - 4^{k+1} = 11 \cdot (11^k - 4^k) + 7 \cdot 4^k$. Imitate Example 2(d).

13. (a) Suppose that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ and $0 \leq k$. Then

$$\sum_{i=0}^{k+1} 2^i = \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1,$$

so the equation still holds for the new value of k.

(c) Yes. $\sum_{i=0}^{0} 2^i = 1 = 2^1 - 1$ initially, so the loop never exits and the invariant is true for every value of k in $\mathbb{N}$.

15. (a) $1 + 3 + \cdots + (2n - 1) = n^2$.

17. (a) Assume $p(k)$ is true. Then $(k+1)^2 + 5(k+1) + 1 = (k^2 + 5k + 1) + (2k + 6)$. Since $k^2 + 5k + 1$ is even by assumption and $2k + 6$ is clearly even, $p(k+1)$ is true.

19. *Hint:* $5^{k+1} - 4(k+1) - 1 = 5(5^k - 4k - 1) + 16k$.

21. *Hints:*

$$\frac{1}{n+2} + \cdots + \frac{1}{2n+2} = \left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}\right)$$

and

$$\frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2}.$$

Alternatively, to avoid induction, let $f(n) = \sum_{i=1}^{n} \frac{1}{i}$ and write both sides in terms of f. The left-hand side is $f(2n) - f(n)$, and the right-hand side is

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) + \cdots + \left(\frac{1}{2n}\right) - 2 \cdot \left[\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right) + \cdots + \left(\frac{1}{2n}\right)\right]$$

$$= f(2n) - 2 \cdot \frac{1}{2} \cdot f(n).$$

23. *Hints:* $5^{k+2} + 2 \cdot 3^{k+1} + 1 = 5(5^{k+1} + 2 \cdot 3^k + 1) - 4(3^k + 1)$. Show that $3^n + 1$ is always even.

25. Here $p(n)$ is the proposition "$|\sin nx| \le n|\sin x|$ for all $x \in \mathbb{R}$." Clearly, $p(1)$ holds. By algebra and trigonometry,

$$|\sin(k+1)x| = |\sin(kx + x)| = |\sin kx \cos x + \cos kx \sin x|$$

$$\le |\sin kx| \cdot |\cos x| + |\cos kx| \cdot |\sin x| \le |\sin kx| + |\sin x|.$$

Now assume $p(k)$ is true and show $p(k+1)$ is true.

Induction and Recursion

Some of the exercises for this section require only the First Principle of Mathematical Induction and are included to provide extra practice. Most of them deal with sequences. You will see a number of applications later in which sequences are not so obvious.

1. Prove $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for $n \in \mathbb{P}$.

2. For $n \in \mathbb{P}$, prove

(a) $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

(b) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

3. Prove that $n^5 - n$ is divisible by 10 for all $n \in \mathbb{P}$.

4. (a) Calculate b_6 for the sequence (b_n) in Example 2.

(b) Use the recursive definition of (a_n) in Example 3 to calculate a_9.

5. Is the First Principle of Mathematical Induction adequate to prove the fact in Exercise 11(b) on page 159? Explain.

6. Recursively define $a_0 = 1$, $a_1 = 2$, and $a_n = \dfrac{a_{n-1}^2}{a_{n-2}}$ for $n \geq 2$.

(a) Calculate the first few terms of the sequence.

(b) Using part (a), guess the general formula for a_n.

(c) Prove the guess in part (b).

7. Recursively define $a_0 = a_1 = 1$ and $a_n = \dfrac{a_{n-1}^2 + a_{n-2}}{a_{n-1} + a_{n-2}}$ for $n \geq 2$. Repeat Exercise 6 for this sequence.

8. Recursively define $a_0 = 1$, $a_1 = 2$, and $a_n = \dfrac{a_{n-1}^2 - 1}{a_{n-2}}$ for $n \geq 2$. Repeat Exercise 6 for this sequence.

9. Recursively define $a_0 = 0$, $a_1 = 1$, and $a_n = \frac{1}{4}(a_{n-1} - a_{n-2} + 3)^2$ for $n \geq 2$. Repeat Exercise 6 for this sequence.

10. Recursively define $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, and $a_n =$

10. Recursively define $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, and $a_n =$
$a_{n-2} + 2a_{n-3}$ for $n \geq 3$.

(a) Calculate a_n for $n = 3, 4, 5, 6, 7$.

(b) Prove that $a_n > \left(\frac{3}{2}\right)^n$ for all $n \geq 1$.

11. Recursively define $a_0 = a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$.

(a) Calculate the first few terms of the sequence.

(b) Prove that all the a_n's are odd.

(c) Prove that $a_n \leq 2^{n-1}$ for all $n \geq 1$.

12. Recursively define $a_0 = 1$, $a_1 = 3$, $a_2 = 5$, and $a_n = 3a_{n-2} + 2a_{n-3}$ for $n \geq 3$.

(a) Calculate a_n for $n = 3, 4, 5, 6, 7$.

(b) Prove that $a_n > 2^n$ for $n \geq 1$.

(c) Prove that $a_n < 2^{n+1}$ for $n \geq 1$.

(d) Prove that $a_n = 2a_{n-1} + (-1)^{n-1}$ for $n \geq 1$.

13. Recursively define $b_0 = b_1 = b_2 = 1$ and $b_n = b_{n-1} + b_{n-3}$ for $n \geq 3$.

(a) Calculate b_n for $n = 3, 4, 5, 6$.

(b) Show that $b_n \geq 2b_{n-2}$ for $n \geq 3$.

(c) Prove the inequality $b_n \geq (\sqrt{2})^{n-2}$ for $n \geq 2$.

14. For the sequence in Exercise 13, show that $b_n \leq \left(\frac{3}{2}\right)^{n-1}$ for $n \geq 1$.

15. Recursively define $\text{SEQ}(0) = 0$, $\text{SEQ}(1) = 1$, and

$$\text{SEQ}(n) = \frac{1}{n} \cdot \text{SEQ}(n-1) + \frac{n-1}{n} \cdot \text{SEQ}(n-2)$$

for $n \geq 2$. Prove that $0 \leq \text{SEQ}(n) \leq 1$ for all $n \in \mathbb{N}$.

16. As in Exercise 15 on page 159, let $\text{SEQ}(0) = 1$ and $\text{SEQ}(n) = \sum_{i=0}^{n-1} \text{SEQ}(i)$ for $n \geq 1$. Prove that $\text{SEQ}(n) = 2^{n-1}$ for $n \geq 1$.

17. Recall the Fibonacci sequence in Example 2(b) defined by

(B) $\text{FIB}(1) = \text{FIB}(2) = 1$,

(R) $\text{FIB}(n) = \text{FIB}(n-1) + \text{FIB}(n-2)$ for $n \geq 3$.

Prove that

$$\text{FIB}(n) = 1 + \sum_{k=1}^{n-2} \text{FIB}(k) \qquad \text{for } n \geq 3.$$

18. The **Lucas sequence** is defined as follows:

(B) $\text{LUC}(1) = 1$ and $\text{LUC}(2) = 3$,

(R) $\text{LUC}(n) = \text{LUC}(n-1) + \text{LUC}(n-2)$ for $n \geq 3$.

(a) List the first eight terms of the Lucas sequence.

(b) Prove that $\text{LUC}(n) = \text{FIB}(n+1) + \text{FIB}(n-1)$ for $n \geq 2$, where FIB is the Fibonacci sequence defined in Exercise 17.

19. Let the sequence T be defined as in Example 2(a) on page 154 by (B) $T(1) = 1$,

(R) $T(n) = 2 \cdot T(\lfloor n/2 \rfloor)$ for $n \geq 2$.

Show that $T(n)$ is the largest integer of the form 2^k with $2^k \leq n$. That is, $T(n) = 2^{\lfloor \log n \rfloor}$, where the logarithm is to the base 2.

20. (a) Show that if T is defined as in Exercise 19, then $T(n)$ is $O(n)$.

(b) Show that if the sequence Q is defined as in Example 2(b) on page 154 by

(B) $Q(1) = 1$,

(R) $Q(n) = 2 \cdot Q(\lfloor n/2 \rfloor) + n$ for $n \geq 2$,

then $Q(n)$ is $O(n^2)$.

(c) Show that, in fact, $Q(n)$ is $O(n \log_2 n)$ for Q as in part (b).

21. Show that if S is defined as in Example 6 on page 157 by (B) $S(0) = 0$, $S(1) = 1$,

(R) $S(n) = S(\lfloor n/2 \rfloor) + S(\lfloor n/5 \rfloor)$ for $n \geq 2$,

then $S(n)$ is $O(n)$.

Answers

1. The First Principle is adequate for this. For the inductive step, use the identity $4n^2-n+8(n+1)-5 = 4n^2+7n+3 = 4(n+1)^2-(n+1)$.

3. Show that n^5-n is always even. Then use the identity $(n+1)^5 = n^5+5n^4+10n^3+10n^2+5n+1$ [from the binomial theorem]. Use the First Principle of induction to show that n^5-n is always divisible by 5. See Example 1 on page 137.

5. Yes. The oddness of a_n depends only on the oddness of a_{n-1}, since $2a_{n-2}$ is even whether a_{n-2} is odd or not.

7. (b) $a_n = 1$ for all $n \in \mathbb{N}$.
   (c) The basis needs to be checked for $n=0$ and $n=1$. For the inductive step, consider $n \geq 2$ and assume $a_k = 1$ for $0 \leq k < n$. Then $a_n = \frac{a_{n-1}^2+a_{n-2}}{a_{n-1}+a_{n-2}} = \frac{1^2+1}{1+1} = 1$. This completes the inductive step, so $a_n = 1$ for all $n \in \mathbb{N}$ by the Second Principle of Induction.

9. (b) $a_n = n^2$ for all $n \in \mathbb{N}$.
   (c) The basis needs to be checked for $n=0$ and $n=1$. For the inductive step, consider $n \geq 2$ and assume that $a_k = k^2$ for $0 \leq k < n$. To complete the inductive step, note that

$$a_n = \frac{1}{4}(a_{n-1} - a_{n-2} + 3)^2 = \frac{1}{4}[(n-1)^2 - (n-2)^2 + 3]^2 = \frac{1}{4}[2n]^2 = n^2.$$

11. (b) The basis needs to be checked for $n = 0$, 1, and 2. For the inductive step, consider $n \geq 3$ and assume that a_k is odd for $0 \leq k < n$. Then a_{n-1}, a_{n-2}, and a_{n-3} are all odd. Since the sum of three odd integers is odd [if not obvious, prove it], a_n is also odd.
   (c) Since the inequality is claimed for $n \geq 1$ and since you will want to use the identity $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ in the inductive step, you will need $n-3 \geq 1$ in the inductive step. So check the basis for $n = 1$, 2, and 3. For the inductive step, consider $n \geq 4$ and assume that $a_k \leq 2^{k-1}$ for $1 \leq k < n$. To complete the inductive step, note that

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} < 2^{n-2} + 2^{n-3} + 2^{n-4} = \frac{7}{8} \cdot 2^{n-1} < 2^{n-1}.$$

15. Check for $n = 0$ and 1 before applying induction. It may be simpler to prove "SEQ$(n) \leq 1$ for all n" separately from "SEQ$(n) \geq 0$ for all n." For example, assume that $n \geq 2$ and that SEQ$(k) \leq 1$ for $0 \leq k < n$. Then

$$\text{SEQ}(n) = (1/n) * \text{SEQ}(n-1) + ((n-l)/n) * \text{SEQ}(n-2) \leq (1/n) + ((n-l)/n) = 1.$$

The proof that SEQ$(n) \geq 0$ for $n \geq 0$ is almost the same.

17. The First Principle of Induction is enough. Use (R) to check for $n = 3$. For the inductive step from n to $n+1$,

$$\text{FIB}(n+1) = \text{FIB}(n) + \text{FIB}(n-1) = 1 + \sum_{k=1}^{n-2} \text{FIB}(k) + \text{FIB}(n-1) = 1 + \sum_{k=1}^{n-1} \text{FIB}(k).$$

13. (a) 2, 3, 4, 6.
   (b) The inequality must be checked for $n = 3$, 4, and 5 before applying the Second Principle of Mathematical Induction on page 167 to $b_n = b_{n-1} + b_{n-3}$. For the inductive step, consider $n \geq 6$ and assume $b_k \geq 2b_{k-2}$ for $3 \leq k < n$. Then

$$b_n = b_{n-1} + b_{n-3} \geq 2b_{n-3} + 2b_{n-5} = 2b_{n-2}.$$

   (c) The inequality must be checked for $n = 2$, 3, and 4. Then use the Second Principle of Mathematical Induction and part (b). For the inductive step, consider $n \geq 5$ and assume $b_k \geq (\sqrt{2})^{k-2}$ for $2 \leq k < n$. Then

$$b_n = b_{n-1} + b_{n-3} \geq 2b_{n-3} + b_{n-3} = 3b_{n-3} \geq 3(\sqrt{2})^{n-5}$$
$$> (\sqrt{2})^3(\sqrt{2})^{n-5} = (\sqrt{2})^{n-2}.$$

   Note that $3 > (\sqrt{2})^3 \approx 2.828$. This can also be proved without using part (b):

$$b_n \geq (\sqrt{2})^{n-3} + (\sqrt{2})^{n-5} = (\sqrt{2})^{n-2} \cdot \left[\frac{1}{\sqrt{2}} + \frac{1}{2^{3/2}}\right] > (\sqrt{2})^{n-2}.$$

19. For $n > 0$, let $L(n)$ be the largest integer 2^k with $2^k \leq n$. Show that $L(n) = T(n)$ for all n by showing first that $L(\lfloor n/2 \rfloor) = L(n/2)$ for $n \geq 2$ and then using the Second Principle of Induction.

21. Show that $S(n) \leq n$ for every n by the Second Principle of Induction.

Matrices

1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 3 & -2 & 3 \\ 2 & 0 & 1 \end{bmatrix}.$$

Evaluate

(a) a_{11} (b) a_{13} (c) a_{31} (d) $\sum_{i=1}^{3} a_{ii}$

2. Consider the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 0 & 1 & 2 \\ 2 & -1 & 4 & 1 \\ 0 & -3 & 1 & 3 \end{bmatrix}.$$

Evaluate

(a) b_{12} (b) b_{21} (c) b_{23} (d) $\sum_{i=1}^{4} b_{ii}$

3. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 3 & -2 \\ 4 & 2 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 6 & 8 & 5 \\ 4 & -2 & 7 \\ 3 & 1 & 2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 3 \\ 2 & -4 \\ 5 & -2 \end{bmatrix}.$$

Calculate the following when they exist.

(a) $\mathbf{A}^T$ (b) $\mathbf{C}^T$ (c) $\mathbf{A}+\mathbf{B}$
(d) $\mathbf{A}+\mathbf{C}$ (e) $(\mathbf{A}+\mathbf{B})^T$ (f) $\mathbf{A}^T+\mathbf{B}^T$
(g) $\mathbf{B}+\mathbf{B}^T$ (h) $\mathbf{C}+\mathbf{C}^T$

4. For the matrices in Exercise 3, calculate the following when they exist.

(a) $\mathbf{A}+\mathbf{A}$ (b) $2\mathbf{A}$
(c) $\mathbf{A}+\mathbf{A}+\mathbf{A}$ (d) $4\mathbf{A}+\mathbf{B}$

5. Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be matrices in $\mathfrak{M}_{4,3}$ defined by $a_{ij} = (-1)^{i+j}$ and $b_{ij} = i + j$. Find the following matrices when they exist.

(a) $\mathbf{A}^T$ (b) $\mathbf{A}+\mathbf{B}$ (c) $\mathbf{A}^T+\mathbf{B}$
(d) $\mathbf{A}^T+\mathbf{B}^T$ (e) $(\mathbf{A}+\mathbf{B})^T$ (f) $\mathbf{A}+\mathbf{A}$

6. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices in $\mathfrak{M}_{3,3}$ defined by $\mathbf{A}[i,j] = ij$ and $\mathbf{B}[i,j] = i + j^2$.

(a) Find $\mathbf{A}+\mathbf{B}$.

(b) Calculate $\sum_{i=1}^{3} \mathbf{A}[i,i]$.

(c) Does $\mathbf{A}$ equal its transpose $\mathbf{A}^T$?

(d) Does $\mathbf{B}$ equal its transpose $\mathbf{B}^T$?

7. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}.$$

Calculate the following

(a) $\mathbf{AB}$ (b) $\mathbf{BA}$
(c) $\mathbf{A}^2 = \mathbf{AA}$ (d) $\mathbf{B}^2$

8. (a) For the matrices in Exercise 7, calculate

$$(\mathbf{A}+\mathbf{B})^2 \quad \text{and} \quad \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2.$$

(b) Are the answers to part (a) the same? Discuss.

9. (a) List all the 3×3 matrices whose rows are

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

(b) Which matrices obtained in part (a) are equal to their transposes?

10. In this exercise, $\mathbf{A}$ and $\mathbf{B}$ represent matrices. True or false?

(a) $(\mathbf{A}^T)^T = \mathbf{A}$ for all $\mathbf{A}$.

(b) If $\mathbf{A}^T = \mathbf{B}^T$, then $\mathbf{A} = \mathbf{B}$.

(c) If $\mathbf{A} = \mathbf{A}^T$, then $\mathbf{A}$ is a square matrix.

(d) If $\mathbf{A}$ and $\mathbf{B}$ are the same size, then $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

11. For each $n \in \mathbb{N}$, let

$$\mathbf{A}_n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B}_n = \begin{bmatrix} 1 & (-1)^n \\ -1 & 1 \end{bmatrix}.$$

(a) Give $\mathbf{A}_n^T$ for all $n \in \mathbb{N}$.

(b) Find $\{n \in \mathbb{N} : \mathbf{A}_n^T = \mathbf{A}_n\}$.

(c) Find $\{n \in \mathbb{N} : \mathbf{B}_n^T = \mathbf{B}_n\}$.

(d) Find $\{n \in \mathbb{N} : \mathbf{B}_n = \mathbf{B}_0\}$.

12. For $\mathbf{A}$ and $\mathbf{B}$ in $\mathfrak{M}_{m,n}$, let $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$. Show that

(a) $(\mathbf{A}-\mathbf{B})+\mathbf{B} = \mathbf{A}$

(b) $-(\mathbf{A}-\mathbf{B}) = \mathbf{B}-\mathbf{A}$

(c) $(\mathbf{A}-\mathbf{B})-\mathbf{C} \neq \mathbf{A}-(\mathbf{B}-\mathbf{C})$ in general

13. Consider $\mathbf{A}$, $\mathbf{B}$ in $\mathfrak{M}_{m,n}$ and a, b, c in $\mathbb{R}$. Show that

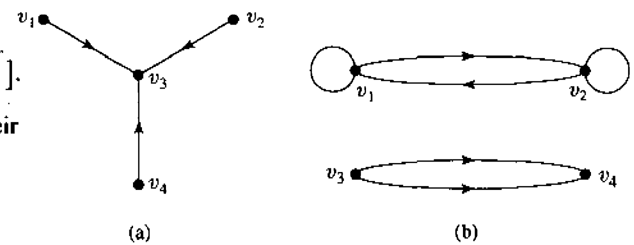
(a) $c(a\mathbf{A} + b\mathbf{B}) = (ca)\mathbf{A} + (cb)\mathbf{B}$

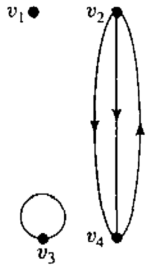
(b) $-a\mathbf{A} = (-a)\mathbf{A} = a(-\mathbf{A})$

(c) $(a\mathbf{A})^T = a\mathbf{A}^T$

14. Prove parts (b), (c), and (d) of the theorem on page 108.

15. Give the matrices for the digraphs in Figure 3.





(c)

Figure 3 ▲

16. Write matrices for the graphs in Figure 4.

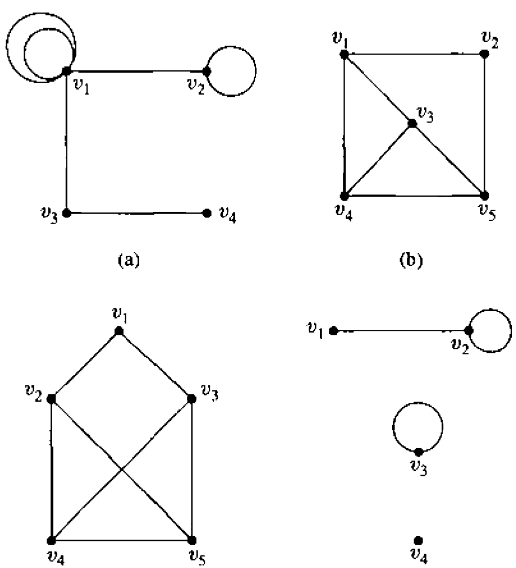


Figure 4 ▲

17. For each matrix in Figure 5, draw a digraph having the matrix.

$$\begin{bmatrix} 0 & 0 & 2 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (b) (c)

Figure 5 ▲

18. For each matrix in Figure 6, draw a graph having the matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

(a) (b)

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(c) (d)

Figure 6 ▲

19. Give a matrix for each of the relations in Exercise 1 on page 99.

20. Draw a digraph having the matrix in Figure 6(b).

21. Give a matrix for each of the relations in Exercise 2 on page 99.

Answers

1. (a) 1. (c) 2.

3. (a) $\begin{bmatrix} -1 & 1 & 4 \\ 0 & 3 & 2 \\ 2 & -2 & 3 \end{bmatrix}$. (c) $\begin{bmatrix} 5 & 8 & 7 \\ 5 & 1 & 5 \\ 7 & 3 & 5 \end{bmatrix}$.

(e) $\begin{bmatrix} 5 & 5 & 7 \\ 8 & 1 & 3 \\ 7 & 5 & 5 \end{bmatrix}$. (g) $\begin{bmatrix} 12 & 12 & 8 \\ 12 & -4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$.

5. (a) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$. (c) Not defined.

(e) $\begin{bmatrix} 3 & 2 & 5 & 4 \\ 2 & 5 & 4 & 7 \\ 5 & 4 & 7 & 6 \end{bmatrix}$.

7. (a) $\begin{bmatrix} -15 & 45 \\ -5 & 15 \end{bmatrix}$. (c) $\begin{bmatrix} 18 & 54 \\ 6 & 18 \end{bmatrix}$.

9. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

11. (a) $\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$.

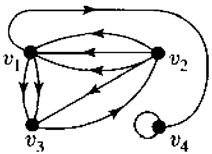
(c) $\{n \in \mathbb{N} : n \text{ is odd}\}$.

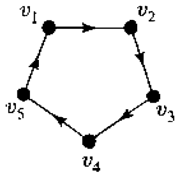
13. (a) The (i, j) entry of $a\mathbf{A}$ is $a\mathbf{A}[i, j]$. Similarly for $b\mathbf{B}$, and so the (i, j) entry of $a\mathbf{A} + b\mathbf{B}$ is $a\mathbf{A}[i, j] + b\mathbf{B}[i, j]$. So the (i, j) entry of $c(a\mathbf{A} + b\mathbf{B})$ is $ca\mathbf{A}[i, j] + cb\mathbf{B}[i, j]$. A similar discussion shows that this is the (i, j) entry of $(ca)\mathbf{A} + (cb)\mathbf{B}$. Since their entries are equal, the matrices $c(a\mathbf{A} + b\mathbf{B})$ and $(ca)\mathbf{A} + (cb)\mathbf{B}$ are equal.

(c) The (j, i) entries of both $(a\mathbf{A})^T$ and $a\mathbf{A}^T$ equal $a\mathbf{A}[i, j]$. Here $1 \le i \le m$ and $1 \le j \le n$. So the matrices are equal.

15. (a) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

17. (a) 

(c) 

19. (a) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(c) See Example 5(a) on page 110.

(e) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

21. (a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(e) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(g) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(i) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Multiplication of Matrices

1. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ -2 & 3 \end{bmatrix}.$$

Find the following when they exist.

(a) $\mathbf{AB}$ (b) $\mathbf{BA}$ (c) $\mathbf{ABA}$

(d) $\mathbf{A} + \mathbf{B}^T$ (e) $3\mathbf{A}^T - 2\mathbf{B}$ (f) $(\mathbf{AB})^2$

2. Let

$$\mathbf{C} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and let **A** and **B** be as in Exercise 1. Find the following when they exist.

(a) $\mathbf{AC}$ (b) $\mathbf{BC}$ (c) $\mathbf{C}^2$

(d) $\mathbf{C}^T\mathbf{C}$ (e) $\mathbf{CC}^T$ (f) $73\mathbf{C}$

3. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -4 & 3 & 1 \\ 2 & 0 & 1 & -2 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

$$\text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Find the following when they exist.

(a) $\mathbf{A}^2$ (b) $\mathbf{B}^2$ (c) $\mathbf{AB}$ (d) $\mathbf{BA}$

4. Let **A** and **B** be as in Exercise 3. Find the following when they exist.

(a) $\mathbf{BA}^T$ (b) $\mathbf{A}^T\mathbf{B}$

(c) $5(\mathbf{AB})^T - 3\mathbf{B}^T\mathbf{A}^T$

5. (a) Calculate both $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ for

$$\mathbf{A} = \begin{bmatrix} -1 & 4 \\ 2 & 5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$\text{and} \quad \mathbf{C} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}.$$

(b) Calculate both $\mathbf{B}(\mathbf{AC})$ and $(\mathbf{BA})\mathbf{C}$.

6. Let **A**, **B**, and **C** be as in Exercise 5. Calculate:

(a) both **AB** and **BA**

(b) both **AC** and **CA**

(c) $\mathbf{A}^2$

7. Let

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$\text{and} \quad \mathbf{C} = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}.$$

(a) Calculate $\mathbf{A}(\mathbf{BC})$ and $(\mathbf{AB})\mathbf{C}$.

(b) Calculate $\mathbf{A}(\mathbf{B}^2)$ and $(\mathbf{AB})\mathbf{B}$.

8. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Calculate

(a) $\mathbf{A} * \mathbf{A}$ (b) $\mathbf{A} * \mathbf{A} * \mathbf{A}$

(c) $\mathbf{A} * \mathbf{A} * \cdots * \mathbf{A}$ for 17 factors

9. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Calculate

(a) $\mathbf{A} * \mathbf{A}$ (b) $\mathbf{A} * \mathbf{A} * \mathbf{A}$

(c) $\mathbf{A} * \mathbf{A} * \cdots * \mathbf{A}$ for 72 factors

10. Draw a digraph associated with the matrix **A** of

(a) Exercise 8. (b) Exercise 9.

11. Let **M** be the adjacency matrix for the digraph in Example 1. One can check that $\mathbf{M}^2 = \begin{bmatrix} 2 & 7 & 1 & 2 \\ 0 & 4 & 0 & 0 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix}$.

Use $\mathbf{M}^2$ to find the number of paths of length 2

(a) from v_1 to itself. (b) from v_1 to v_3.

(c) from v_1 to v_4. (d) from v_2 to v_1.

12. Let **A** be the Boolean matrix for the digraph in Example 1 and Exercise 11. Use $\mathbf{A} * \mathbf{A}$ to determine whether there are or are not paths of length 2

(a) from v_1 to itself. (b) from v_1 to v_3.

(c) from v_1 to v_4. (d) from v_2 to v_1.

13. (a) Calculate $\mathbf{M}^3$ for the adjacency matrix in Example 1.

(b) Find the number of paths of length 3 from v_3 to v_2.

(c) List the paths of length 3 from v_3 to v_2 using the labeling of Figure 2(b).

14. This exercise refers to the graph in Example 3.

(a) Draw the graph; just remove the arrowheads from Figure 2(a). Label the edges as in Figure 2(b).

(b) How many paths of length 2 are there from v_3 to itself?

(c) List the paths from v_3 to itself of length 2.

(d) How many paths of length 3 are there from v_3 to itself?

(e) List the paths from v_3 to itself of length 3.

15. Repeat parts (a) to (d) of Exercise 14 for the vertex v_2.

16. Show that a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse if and only if $ad - bc \neq 0$, in which case the inverse is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Hint: Try to solve $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for x, y, z, and w.

17. Use Exercise 16 to determine which of the following matrices have inverses. Find the inverses when they exist and check your answers.

(a) $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (d) $\mathbf{C} = \begin{bmatrix} 2 & -3 \\ 5 & 8 \end{bmatrix}$

(e) $\mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

18. Find 2×2 matrices that show that $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$ does not generally hold.

19. Show that if $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix, then $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$. Note that both sides of the equality represent $p \times m$ matrices.

20. (a) Prove the cancellation law for $\mathfrak{M}_{m,n}$ under addition; i.e., prove that if $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are in $\mathfrak{M}_{m,n}$ and $\mathbf{A}+\mathbf{C} = \mathbf{B}+\mathbf{C}$, then $\mathbf{A} = \mathbf{B}$.

(b) Show that the cancellation law for $\mathfrak{M}_{n,n}$ under multiplication fails; i.e., show that $\mathbf{AC} = \mathbf{BC}$ need not imply $\mathbf{A} = \mathbf{B}$ even when $\mathbf{C} \neq \mathbf{0}$.

21. (a) Let $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for some fixed a in $\mathbb{R}$. Show that $\mathbf{AB} = \mathbf{BA}$ for all $\mathbf{B}$ in $\mathfrak{M}_{2,2}$.

(b) Consider a fixed matrix $\mathbf{A}$ in $\mathfrak{M}_{2,2}$ that satisfies $\mathbf{AB} = \mathbf{BA}$ for all $\mathbf{B}$ in $\mathfrak{M}_{2,2}$. Show that

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \quad \text{for some} \quad a \in \mathbb{R}.$$

Hint: Write $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and try $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

22. (a) Show directly that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ for matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in $\mathfrak{M}_{2,2}$.

(b) Did you enjoy part (a)? If yes, give a direct proof of the general associative law for matrices.

23. (a) Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrices and let $\mathbf{C}$ be an $n \times p$ matrix. Show that the distributive law holds: $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$.

(b) Verify the distributive law $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$. First specify the sizes of the matrices for which this makes sense.

24. Show that if $\mathbf{A}$ is an $m \times n$ matrix, then

(a) $\mathbf{I}_m\mathbf{A} = \mathbf{A}$. (b) $\mathbf{AI}_n = \mathbf{A}$.

Answers

1. (a) $\begin{bmatrix} -8 & 13 \\ 2 & 9 \end{bmatrix}$. (c) $\begin{bmatrix} 31 & -16 & -6 \\ 29 & 4 & 26 \end{bmatrix}$. (e) $\begin{bmatrix} -1 & 7 \\ 8 & 0 \\ 16 & 0 \end{bmatrix}$.

3. The products written in parts (a) and (c) do not exist.

5. (a) $\begin{bmatrix} 1 & 10 \\ 11 & 19 \end{bmatrix}$.

7. (a) $\begin{bmatrix} 7 & 14 \\ 8 & 11 \\ 2 & -6 \end{bmatrix}$.

9. (a) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

11. (a) 2. (c) 2.

13. (a) $\mathbf{M}^3 = \begin{bmatrix} 3 & 20 & 2 & 3 \\ 0 & 8 & 0 & 0 \\ 2 & 9 & 1 & 2 \\ 0 & 4 & 0 & 0 \end{bmatrix}$.

(c) $f\,a\,b$, $f\,a\,c$, $f\,b\,d$, $f\,b\,e$, $f\,c\,d$, $f\,c\,e$, $f\,h\,j$, $k\,j\,d$, $k\,j\,e$.

15. (a) Simply remove the arrows from Figure 2 on page 440.

(c) $d\,d$, $e\,e$, $d\,e$, $e\,d$, $b\,b$, $c\,c$, $b\,c$, $c\,b$, $j\,j$.

17. (a) $\mathbf{I}^{-1} = \mathbf{I}$. (c) Not invertible.

19. For $1 \le k \le p$ and $1 \le i \le m$,

$$(\mathbf{B}^T\mathbf{A}^T)[k,i] = \sum_{j=1}^{n} \mathbf{B}^T[k,j]\mathbf{A}^T[j,i] = \sum_{j=1}^{n} \mathbf{B}[j,k]\mathbf{A}[i,j].$$

Compare with the (k,i)-entry of $(\mathbf{AB})^T$.

21. (a) In fact, $\mathbf{AB} = \mathbf{BA} = a\mathbf{B}$ for all $\mathbf{B}$ in $\mathfrak{M}_{2,2}$.

(b) $\mathbf{AB} = \mathbf{BA}$ with $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ forces $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, so $b = c = 0$. So $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. Now try $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

23. (a) Consider $1 \le i \le m$ and $1 \le k \le p$, and compare the (i,k) entries of $(\mathbf{A}+\mathbf{B})\mathbf{C}$ and $\mathbf{AC}+\mathbf{BC}$.

(b) If $\mathbf{A}$ is $m \times n$, then $\mathbf{B}$ and $\mathbf{C}$ must both be $n \times r$ for the same r. For $1 \le i \le m$ and $1 \le k \le r$, show $(\mathbf{A}(\mathbf{B}+\mathbf{C}))[i,k] = (\mathbf{AB}+\mathbf{AC})[i,k]$.